Question 1

(i) First part bookwork.

(ii) Let $Tx = \langle f_\gamma(x) \rangle_{\gamma \in \Gamma}$. If T is bounded, then there exists M such that $\sup_{\gamma \in \Gamma} |f_\gamma(x)| \leq M \|x\|$ for each $x \in X$. In particular $|f_\gamma(x)| \leq M$ for each $x \in X$. So $\{f_\gamma : \gamma \in \Gamma\} \subseteq X^*$ and further $\|f_\gamma\|_\infty \leq M$ for each $\gamma \in \Gamma$, so this set is bounded.

Take $\alpha = \sup_{\gamma \in \Gamma} \|f_\gamma\|_\infty$. It is clear that:

$$\|Tx\| = \sup_{\gamma \in \Gamma} |f_\gamma(x)| \leq \alpha \|x\|$$

Now note that for each $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that $\alpha - \varepsilon/2 < \|f_\gamma\|_\infty \leq \alpha$. Then there exists $\|e_\gamma\| = 1$ in X such that $\alpha \geq |f_\gamma e_\gamma| > \alpha - \varepsilon$. So $(\alpha - \varepsilon)\|e_\gamma\| < \|Te_\gamma\| \leq \alpha \|e_\gamma\|$. Since ε was arbitrary we have $\|T\| = \alpha$.

Conversely, if $\{f_\gamma : \gamma \in \Gamma\}$ is bounded and we have $\|Tx\| = \sup_{\gamma \in \Gamma} |f_\gamma(x)| \leq \alpha \|x\|$ as in the previous case. So T is bounded.

(iii) Pick $\Gamma = X$ and take $\langle f_x \rangle_{x \in X}$ to be the support functional at x. Then $\{f_x : x \in X\} \subseteq X^*$ is bounded and so $Ty = \langle f_x(y) \rangle_{n \in \mathbf{N}}$ is continuous. For each $y \in X$ we have $|f_x(y)| \leq \|y\|$ with equality if $y = x$. Then $\|Ty\| = \sup_{x \in X} |f_x(y)| = \|y\|$. So T is an isometric embedding.

If X is separable, take $\Gamma = \mathbf{N}$, a dense subset $\langle e_n \rangle_{n \in \mathbf{N}}$ and the corresponding support functionals $\langle f_n \rangle_{n \in \mathbf{N}}$. We have $\|Tx\| = \sup_{n \in \mathbf{N}} |f_n(x)|$. We have $|f_n(e_n)| \leq \|e_n\|$ for each e_n, while $|f_n(x)| \leq \|x\|$ for all $x \in X$. So we have $\|Te_n\| = \|e_n\|$ for each n. Since $\langle e_n \rangle_{n \in \mathbf{N}}$ is dense, we have $\|Tx\| = \|x\|$ for all $x \in X$ by continuity.

If $X = Y^*$ for separable Y, take $\langle x_n \rangle_{n \in \mathbf{N}}$ a dense subset of B_Y and consider the bounded subset $\{\widehat{x_n} : n \in \mathbf{N}\} \subseteq Y^{**} = X^*$. Then define $Tf = \langle f(x_n) \rangle_{n \in \mathbf{N}}$. For each $f \in X$ and $\varepsilon > 0$ there exists $y \in S_Y$ such that $\|f\|_\infty - \varepsilon/2 < |f(y)| \leq \|f\|_\infty$. In particular, by approximating y there exists some n such that $\|f\|_\infty - \varepsilon < |f(x_n)| \leq \|f\|_\infty$. It follows that $\|Tf\| = \sup_{n \in \mathbf{N}} |f(x_n)| = \|f\|_\infty$.

(iv) Take $T \in B(Y, \ell_\infty(\Gamma))$. Then $Ty = \langle f_\gamma(y) \rangle_{\gamma \in \Gamma}$ for each $y \in Y$. By part ii, each $f_\gamma : Y \to \mathbf{C}$ is in Y^*, and since $\|T\| = \sup_{\gamma \in \Gamma} \|f_\gamma\|_\infty$, we have $\|f_\gamma\|_\infty \leq \|T\|$ for each $\gamma \in \Gamma$. Applying Hahn–Banach to each f_γ, there exists $\widetilde{f_\gamma} : Z \to \mathbf{C}$ such that $\|\widetilde{f_\gamma}\| \leq \|T\|$. Then $\widetilde{T} = \langle \widetilde{f_\gamma} \rangle_{\gamma \in \Gamma}$ works and satisfies $\|\widetilde{T}\| \leq \|T\|$.

(v) *If*: If the condition holds, then there exists an isometric embedding $T : X \to \ell_\infty(\Gamma)$ for a set Γ. Then there exists a bounded linear map $P : \ell_\infty(\Gamma) \to X$ such that $\|P\| \leq \lambda$ and $P \circ T$ is the identity on X. Now let Y be a subspace of a normed space Z and let $S : Y \to X$ be a bounded linear map. Then $T \circ S : Y \to \ell_\infty(\Gamma)$ is bounded. We have shown that $\ell_\infty(\Gamma)$ is 1–injective, so there exists $\widetilde{T \circ S} : Z \to \ell_\infty(\Gamma)$ such that $\widetilde{T \circ S}$ restricts to $T \circ S$ and $\left\|\widetilde{T \circ S}\right\| \leq \|T \circ S\|$. Then $P \circ \widetilde{T \circ S}$ restricts to $P \circ (T \circ S) = S$. Finally we have, since T is an isometry:

$$\left\|P \circ \widetilde{T \circ S}\right\| \leq \left\|\widetilde{T \circ S}\right\| \leq \lambda \|T \circ S\| = \lambda \|S\|$$

So X is λ–injective.

Only if: Suppose that X is λ–injective and let $T : X \to Z$ be an isometric embedding. Then take $Y = T(x)$. We can take the bounded inverse $T^{-1} : T(X) \to X$. Then there exists $\widetilde{T^{-1}} \in B(Z, X)$ such that $\widetilde{T^{-1}}$ restricts to T^{-1} with $\left\|\widetilde{T^{-1}}\right\| \leq \lambda \left\|T^{-1}\right\|$. Take $P = \widetilde{T^{-1}}$. Then $P \circ T$ is the identity since P restricts to T^{-1} on Y. Further, $\left\|\widetilde{T^{-1}}\right\| \leq \lambda \left\|T^{-1}\right\| = \lambda$.

1 Question 2

Part a

Bookwork up to last part, which is also partly bookwork.

Suppose $\{f_n : n \in \mathbf{N}\} \subseteq B_{X^*}$ separates the points of X. Let σ be the initial topology induced by the $\{f_n : n \in \mathbf{N}\} \subseteq B_{X^*}$ and let K be a w–compact set. It is bookwork to prove that σ is metrizable. Then the inclusion $(K, w) \to (K, \sigma)$ is a continuous bijection from a compact space to a Hausdorff space, and hence $(K, w) = (K, \sigma)$. That is, (K, w) is metrizable.

Part b

Part bi

Let $\langle x_n \rangle_{n \in \mathbf{N}}$ be a sequence in a Banach space X that is weakly convergent to x. Then $\{x_n : n \in \mathbf{N}\} \cup \{x\}$ is weakly compact, and hence is norm bounded by the bookwork in part a. We deduce that any weakly convergent sequence is norm bounded.

Since $\langle f_n \rangle_{n \in \mathbf{N}}$ converges weakly to 0, it is a norm–bounded sequence, say $\|f_n\|_\infty \leq M$. Since point evaluations are bounded linear functionals, we also have that $f_n \to 0$ pointwise. Then applying the dominated convergence theorem we have $\|f_n\|_1 \to 0$.

Part bii

Let $\alpha = \inf_{y \in C} \|y\|$. For each $n \in \mathbf{N}$, pick $x_n \in C$ such that $\alpha < \|x_n\| < \alpha + \frac{1}{n}$. Let $Y = \overline{\operatorname{span}\{x_n : n \in \mathbf{N}\}}$. Since Y is a closed subspace of a reflexive space, it is reflexive. That is, (B_Y, w) is compact.

Y is also separable as the closed linear span of a countable set. Take a dense countable subset $\langle e_n \rangle_{n \in \mathbf{N}}$. Then the countable subset $\langle \widehat{e_n} \rangle_{n \in \mathbf{N}} \subseteq B_{Y^{**}}$ separates the points of Y^*. So the topology on any weakly compact subset of Y^* is metrizable. In particular, (B_{Y^*}, w) is metrizable. Since Y is reflexive we have $(B_{Y^*}, w) = (B_{Y^*}, w^*)$. So (B_{Y^*}, w) is compact and metrizable, and hence must be weakly separable.

Note that if $\mathcal{S} = \{z_n : n \in \mathbf{N}\}$ is w^*–dense in B_{Y^*}, it must separate the points of Y: if $f, g \in B_{Y^*}$ had $f(z_n) = g(z_n)$ for each n, we would obtain that $f = g$ from the continuity of f and g. It follows that every weakly compact subset of Y is metrizable, in particular (B_Y, w) is metrizable, hence sequentially compact. (since we already know it to be compact)

By construction, $\langle x_n \rangle_{n \in \mathbf{N}}$ is norm (hence weakly) bounded and hence is contained in some ball $MB_Y \cap C$. This set is a w–closed subset of B_Y, and so is metrizable and compact, hence sequentially compact. So some subsequence $\langle x_{n_j} \rangle_{j \in \mathbf{N}}$ converges weakly to $x \in MB_Y \cap C$.

Then:

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\| = \alpha$$

But since $\alpha = \inf_{y\in C} \|y\|$, we must therefore have $\|x\| = \inf_{y\in C} \|y\|$.

Part biii

Let K be a w–compact subset of ℓ_∞. Then the coordinate functionals sending $\langle x_n \rangle_{n\in\mathbf{N}}$ to x_n separate the points of ℓ_∞. So K is metrizable. Then K is w–separable, with countable dense subset $\{z_n : n \in \mathbf{N}\}$. Then $\overline{\operatorname{span} K} = \overline{\operatorname{span}\{z_n : n \in \mathbf{N}\}}^w = \overline{\operatorname{span}\{z_n : n \in \mathbf{N}\}}$ by Mazur. So $\operatorname{span} K$ is norm–separable, and hence so is K as a metric subspace.

Question 3

Part a

Example sheets

Part b

Bookwork

Part c

Bookwork: prove that if X is separable then (B_{X^*}, w^*) is metrizable.

Since X^* is separable, $(B_{X^{**}}, w^*)$ is metrizable and hence sequentially compact by Banach–Alaoglu. A norm–bounded sequence $\langle x_n \rangle_{n\in\mathbf{N}}$ in X remains norm (hence w^*) bounded in X^{**}. So there exists $\phi \in X^{**}$ such that $\widehat{x_n} \xrightarrow{w^*} \phi$. That is, $\phi(f) = \lim_{n\to\infty} \widehat{x_n}(f) = \lim_{n\to\infty} f(x_n)$ for each $f \in X^*$.

If X is not reflexive, there exists $\phi \in B_{X^{**}} \setminus B_X$. Since $(B_{X^{**}}, w^*)$ is metrizable and closed, it is sequentially closed. Since B_X is w^*–dense in $B_{X^{**}}$ by Goldstine, there exists a sequence $\langle x_n \rangle_{n\in\mathbf{N}}$ in B_X (hence norm bonuded) with $\widehat{x_n} \xrightarrow{w^*} \phi$. If some subsequence of $\langle x_n \rangle_{n\in\mathbf{N}}$, say $\langle x_{n_j} \rangle_{j\in\mathbf{N}}$ converged weakly to x in X, we would have $\widehat{x_{n_j}} \xrightarrow{w^*} \widehat{x}$ in X^{**}, (since the embedding is w–to–w^* continuous) hence $\widehat{x} = \phi$, contrary to our assumption that $\phi \notin X$. So $\langle x_n \rangle_{n\in\mathbf{N}}$ is our desired sequence.

Question 4

Part a

Bookwork.

Part b

Bookwork up to $\exp(x) = \cdots$. WLOG take A to be commutative, by taking the maximal commutative subalgebra containing x. Then define $\exp : U \to \mathbf{C}$ by $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, a holomorphic function that is the uniform limit of the polynomials $P_N(z) = \sum_{n=0}^{N} \frac{z^n}{n!}$.

Then we have:

$$\begin{aligned} \exp(x) &:= \Theta_x(f) \\ &= \Theta_x(\lim_{n\to\infty} P_N) \\ &= \lim_{N\to\infty} \Theta_x(P_N) \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} \end{aligned}$$

where we have used that Θ_x is continuous and sends polynomials to the analogy in the Banach algebra. So we get $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Bookwork proving form of spectrum.

From the previous part we have $\sigma_A(f(x)) = \{f(\lambda) : \lambda \in \sigma_A(x)\} \subseteq V$ since $\sigma_A(x) \subseteq U$ and $f(U) \subseteq V$. Consider the HFCs $\Theta_x : \mathcal{O}(V) \to A$ and $\Theta_{f(x)} : \mathcal{O}(V) \to A$. We aim to show that $g \mapsto \Theta_x(g \circ f)$ and $g \mapsto \Theta_{f(x)}(g)$ both satisfy the conditions for the HFC and so must be equal.

They are certainly both homomorphisms and $g \mapsto \Theta_x(g \circ f)$ is continuous as the composition of continuous functions. Further, both maps are unital since $\Theta_x(1 \circ f) = \Theta_x(1) = 1 = \Theta_{f(x)}(1)$. Finally we can see that $\Theta_x(\mathrm{id} \circ f) = \Theta_x(f) = f(x) = \Theta_{f(x)}(\mathrm{id})$. Hence we deduce that $\Theta_x(\bullet \circ f) = \Theta_{f(x)}$ by the uniqueness part of the HFC. So for each holomorphic g we have $\Theta_x(g \circ f) = (g \circ f)(x) = \Theta_{f(x)}(g) = g(f(x))$.

Let $U = B_{\|x\|+\varepsilon}(0)$ with ε picked so that $\|x\|+\varepsilon < 1$ and let V be an open subset containing U. Define a logarithm log on $1 - B_{\|x\|+\varepsilon}(0)$ (note that this is still a positive distance from the origin) and let $f : U \to \mathbf{C}$ be defined by $f(z) = \log(1-z)$. Define $g : f(U) \to \mathbf{C}$ to be the exponential $g(z) = \exp(z)$. Then we have $(g \circ f)(x) = (1_U - \mathrm{id})(x) = 1 - x = g(f(x)) = \exp(f(x))$. Letting $y = f(x)$ gives the result $\exp(y) = 1 - x$.

Question 5

Bookwork up to Invariant Subspace Problem.

Let λ_1 and λ_2 be distinct points in $\sigma(T)$ and fix disjoint open neighborhoods U_1, U_2 thereof in K. Consider the projections $P_1 = P(U_1)$ and $P_2 = P(U_2)$. P_1 is certainly non–zero since $P(U)$ is non–zero for open U. We have $P(U_1)P(U_2) = P(U_1 \cap U_2) = 0$, which means that $P_1 \neq I$. P_1 clearly commutes with every projection as a projection itself, hence we have $TP_1 = P_1 T$. Then $V = \ker(I - P_1) = \mathrm{im}(P_1)$ is a non–trivial invariant subspace.