Question 1

Bookwork and example sheets.

Question 2

[This solution is based off notes taken during Dr. Zsak's example class]

(b) Since $x_n \xrightarrow{w} 0$ we have:

$$0 \in \overline{\operatorname{conv}\{x_n : n \in \mathbf{N}\}}^w = \overline{\operatorname{conv}\{x_n : n \in \mathbf{N}\}}$$

by Mazur. So there exists $p_1 < q_1$ and $t_{p_1}, \ldots, t_{q_1}$ such that $\sum_{i=p_1}^{q_1} t_i = 1$ such that $\left\| \sum_{i=p_1}^{q_1} t_i x_i \right\| < 1$. Then since $0 \in \overline{\operatorname{conv}\{x_n : n \geq q_1\}}^w = \overline{\operatorname{conv}\{x_n : n \geq q_1\}}$, and so on, we can continue this construction to find $p_1 < q_1 < p_2 < q_2 < \ldots$ such that $\sum_{i=p_n}^{q_n} = 1$ and $\left\| \sum_{i=p_n}^{q_n} t_i x_i \right\| < 1/n$. Then $\sum_{i=p_n}^{q_n} t_i x_i \to 0$ as desired.

(c) Since X^* is separable, $B_{X^{**}}$ is w^*–metrizable, hence $B_{X^{**}}$ is w^*–sequentially compact. Since $\langle x_n \rangle_{n \in \mathbf{N}}$ is norm bounded in X, it is norm bounded (hence w^*–bounded) in X^{**}. So there exists some subsequence $\langle \widehat{y_n} \rangle_{n \in \mathbf{N}}$ converging w^* to $\phi \in X^{**}$. That is, for $f \in X^*$ we have $\widehat{y_n}(f) = f(y_n) \to \phi(f)$. So $\langle f(y_n) \rangle_{n \in \mathbf{N}}$ is Cauchy for each $f \in X^*$.

Let u_n be a convex block of the $\langle y_n \rangle_{n \in \mathbf{N}}$, say $u_n = \sum_{i=p_n}^{q_n} t_i y_i$. Then for $f \in X^*$ we have:

$$\begin{aligned} |f(u_n - y_n)| &\leq \sum_{i=p_n}^{q_n} t_i (f(y_i) - f(y_n)) \\ &\leq \sup_{p_n \leq i \leq q_n} |f(y_i) - f(y_n)| \end{aligned}$$

Since $p_n, q_n \to \infty$, we have $|f(u_n - y_n)| \to 0$ for each $f \in X^*$. That is, $u_n - y_n \xrightarrow{w} 0$.

(d) Let $\langle z_n \rangle_{n \in \mathbf{N}}$ be a sequence in B_Z with $z_n \xrightarrow{w} 0$. Pick $\langle w_n \rangle_{n \in \mathbf{N}} \subseteq (4/3) B_X$ such that $q(w_n) = z_n$. Passing to a subsequence, we can assume that $\langle w_n \rangle_{n \in \mathbf{N}}$ w^*–converges to some $\psi \in X^{**}$. Take a convex block $\langle u_n \rangle_{n \in \mathbf{N}}$ of $\langle z_n \rangle_{n \in \mathbf{N}}$ such that $u_n \to 0$. Take the corresponding convex block of $\langle w_n \rangle_{n \in \mathbf{N}}$, $\langle x_n \rangle_{n \in \mathbf{N}}$ so that $q(x_n) = u_n$ for each $n \in \mathbf{N}$. Then by the previous part, we have $x_n - w_n \xrightarrow{w} 0$. We can now conclude, since we have $\|x_n - w_n\| < 3$ and $\|q(x_n - w_n) - z_n\| = \|q(x_n)\| \to 0$.

Question 3

Mostly bookwork. Let $\phi \in L_p^*$. Then there exists $g \in L_q$ such that:

$$\phi(f) = \int f g \, d\mu$$

Then we have $|f_n g| \leq |fg|$. Since $|fg|$ is integrable by Holder's inequality and $f_n g \to 0$ pointwise almost everywhere, we have:

$$\int f_n g \, d\mu \to 0$$

by DCT, and hence $\phi(f_n) \to 0$. So $f_n \xrightarrow{w} 0$.

Question 4

First, any unital $*$–homomorphism $f : A \to B$ between C^*–algebras is continuous: for each $x \in A$ we have $\sigma_B(f(x)) \subseteq \sigma_A(x)$. (since if x is invertible in A, then $f(X)$ is invertible in B) Then $\|f(x)\|^2 = \|(f(x))^*(f(x))\| = r_B(f(x^*x)) \leq r_A(x^*x) = \|x^*x\| = \|x\|^2$, where we have used that if A is normal then $r_A(x) = \|x\|$.

Let A be the maximal commutative C^* subalgebra generated by T. Then $\sigma_A(T) = \sigma(T)$. Then there is a unital $*$–isomorphism $S : \Phi_A \to \sigma(T)$ sending $\phi \in \Phi_A$ to $\phi(T)$. Then the map $f \mapsto f \circ S$ is a unital $*$–isomorphism $C(\sigma(T)) \to C(\Phi_A)$. Then the inverse Gelfand map $G^{-1} : C(\Phi_A) \to A$ is a unital $*$–isomorphism. Finally defining a map $\Gamma : C(\sigma(T)) \to A$ by $\Gamma(f) = G^{-1}(f \circ S)$ we have a unital $*$–isomorphism. First, we have for $\phi \in \Phi_A$: $(G \circ \Gamma)(\text{id})(\phi) = S\phi = \phi(T) = \hat{T}(\phi)$. So $(G \circ \Gamma)(\text{id}) = \hat{T} = G(T)$. Since G is an isomorphism it follows that $T = \Gamma(\text{id})$. Now for $\lambda \in K$, we have $(G \circ \Gamma)(\lambda)(\phi) = G^{-1}(\lambda)(\phi) = \lambda = \widehat{\lambda I}(\phi) = (G(\lambda I))\phi$. So $\Gamma(\lambda) = \lambda I$.

For uniqueness, note that the conditions $\Gamma(\text{id}) = T$ and $\Gamma(\lambda) = \lambda$ together imply that for any polynomial $p \in \mathbf{C}[X, Y]$ we have $\Gamma(p(z, \overline{z})) = p(T, T^*)$. Since the polynomials in $z, \overline{z}$ are dense in $C(\sigma(T))$, we have that Γ is uniquely determined on a dense subset of $C(\sigma(T))$, hence uniquely determined.

Now suppose that $\sigma(T) = K_1 \cup K_2$ is disconnected. Let U_1 and U_2 be open neighborhoods of K_1 and K_2 with positive distance between them. Define $f(z) = 1$ on U_1 and $f(z) = 0$ on U_2. Then f is continuous since it is continuous on both connected components. We have $\Gamma(f)^2 = \Gamma(f^2) = \Gamma(f)$, so $\Gamma(f)$ is a projection, and commutes with A since $\Gamma(f) \in A$. We lastly need to verify that $\Gamma(f)$ is non–trivial. We do this by noting that since Γ is a $*$–isomorphism onto A, it is injective. Since f is not identically 0 or the 1, we have $\Gamma(f) \notin \{0, I\}$. Hence $\Gamma(f)$ is a non–trivial projection commuting with T and we are done.

Question 5

(1) $\implies$ (2): Since T^{**} extends T we have $T^{**}x \in TB_X \subseteq \overline{TB_X}$ for each $x \in X$. Hence:

$$B_X \subseteq (T^{**})^{-1}(\overline{TB_X}) \tag{*}$$

We have that $\overline{TB_X}$ is w–compact. So $\overline{TB_X}$ is w^*–compact, hence w^*–closed. So $(T^{**})^{-1}(\overline{TB_X})$ is w^*–closed. Taking closures on both sides of the inclusion $(*)$ and then applying T^{**} gives $T^{**}(B_{X^{**}}) \subseteq \overline{TB_X} \subseteq Y$. So $T^{**}(X^{**}) = \bigcup_{r=1}^{\infty} T^{**}(rB_{X^{**}}) \subseteq Y$. So we have proved (2).

(2) $\implies$ (3): Suppose that $T^{**}(X^{**}) \subseteq Y$. Then for each $g \in X^{**}$, we have $g \circ T^{**} = T^{**}g \in Y = (Y^*, w^*)^*$. So by the universal property, T^* is w^*–to–w continuous.

(3) $\implies$ (4): Suppose that $T^* : Y^* \to X^*$ is w^*–to–w continuous. Since B_{X^*} is w^*–compact by Banach–Alaoglu, we have that $T^*(B_{X^*})$ is w–compact, in particular w–closed. So we immediately get that T^* is weakly compact.

(4) $\implies$ (1): Suppose that T^* is compact. Using (1) $\implies$ (4), we have that T^{**} is compact. Since T^{**} extends T, we have $T(B_X) \subseteq T^{**}(B_{X^{**}})$. $B_{X^{**}}$ is w^*–compact by Banach–Alaoglu. Since $T^{**} : X^{**} \to Y^{**}$ is w^*–to–w continuous by (1) $\implies$ (3), we have

that $T^{**}(B_{X^{**}})$ is w–compact in Y^{**}. So $\overline{T(B_X)}^w$ is a w–closed subset of the w–compact set $T^{**}(B_{X^{**}})$, so is w–compact.

If X is reflexive, then $X = X^{**}$. Then $T^{**}(X) = T(X) \subseteq Y$, so T is weakly compact. If Y is reflexive, then so is Y^*. Consider $T^* : Y^* \to X^*$ and $T^{***} : Y^* \to X^*$. Then $T^{***}(Y^*) = T^*(Y^*) \subseteq X^*$, so T^* is weakly compact. So T is weakly compact.