Question 1

Pure bookwork.

Question 2

Bookwork omitted.

(d) Suppose $\overline{C}$ is weakly compact in X. Then it is w^* compact, hence w^* closed, in X^{**}. Since $C \subseteq \overline{C}$, it follows that $\overline{C}^{w^*} \subseteq \overline{C} \subseteq X$. (since the w^* topology is smaller than the norm topology, we obtain $\overline{C}^{w^*} = \overline{C}$ in this case)

Now suppose that $\overline{C}^{w^*} \subseteq X$. Since C is bounded, we have that $C \subseteq MB_X$ for some $M > 0$, so $\overline{C}^{w^*}$ is a w^*–closed subset of the w^*–compact set $MB_{X^{**}}$, and hence is w^*–compact itself. But $\overline{C}^{w^*} \subseteq X$, so it is w–compact in X. (take the preimage under the embedding $X \to X \subseteq X^{**}$, which is a w–to–w^* homomorphism) Note that since $\overline{C}$ is norm closed and C is convex, $\overline{C}$ is also weakly closed, so $\overline{C} \subseteq \overline{C}^{w^*}$. Since $\overline{C}$ is then a weakly closed subset of a weakly compact set, it is itself weakly compact.

(e) That dual operators are w^*–to–w^* continuous is on the example sheets. Note from this that T^{**} extends T, since it sends $\widehat{x}$ to $\widehat{Tx}$. Suppose that $T^{**}(X^{**}) \subseteq Y$. Then in particular $T^{**}(B_{X^{**}}) \subseteq Y$, with $T^{**}(B_{X^{**}})$ being w^*–compact since $B_{X^{**}}$ is. Since T^{**} extends T we have $TB_X \subseteq T^{**}(B_{X^{**}})$. Then $\overline{TB_X}^{w^*} \subseteq T^{**}(B_{X^{**}}) \subseteq Y$. So $\overline{TB_X}$ is weakly compact.

Now suppose that $\overline{TB_X}$ is weakly compact then $\overline{TB_X}^{w^*} \subseteq Y$. Recalling again that T^{**} extends T, we can see that $(T^{**})^{-1}(\overline{TB_X}^{w^*}) \supseteq T^{-1}(\overline{TB_X}^{w^*}) \supseteq B_X$. Since $\overline{TB_X}^{w^*}$ is w^*–closed, so is $(T^{**})^{-1}(\overline{TB_X}^{w^*})$. So by taking w^*–closures we have $B_{X^{**}} \subseteq (T^{**})^{-1}(\overline{TB_X}^{w^*})$. So $T^{**}(B_{X^{**}}) \subseteq \overline{TB_X}^{w^*} \subseteq Y$. By scaling and taking unions we get $T^{**}(X^{**}) \subseteq Y$.

Question 3

Bookwork omitted.

Suppose that there existed a Banach space X with $X^* = c_0$ or $X^* = L_1[0,1]$. Then $B_{X^*} = \overline{\text{conv}(\text{Ext}(B_{X^*}))}^{w^*}$. It is therefore enough to show that the closed unit balls of c_0 and $L_1[0,1]$ have no extreme points.

Let $x \in B_{c_0}$. Since $x_n \to 0$, there exists n such that $|x_n| < 1/4$. Then setting $u = x + \frac{1}{4}e_n \neq x$ and $v = x - \frac{1}{4}e_n \neq x$ we have $\frac{1}{2}u + \frac{1}{2}v = x$, so x is not extreme.

Let $f \in C_{L_1[0,1]}$. Then there exists x such that $\int_0^x |f| = \int_x^1 |f| = 1/2$. Then we have:

$$\frac{1}{2}\underbrace{f1_{[0,x]}}_{\neq f} + \frac{1}{2}\underbrace{f1_{[x,1]}}_{\neq f} = f$$

since f cannot be identically zero on either $[0,x]$ and $[x,1]$.

By Banach–Stone, $C[0,1]$ being isometrically isomorphic to $C([0,1]\cup[2,3])$ is an equivalent claim to $[0,1]$ being homeomorphic to $[0,1]\cup[2,3]$. But the former space is connected and the latter disconnected, so they cannot be homeomorphic.

Question 4

Pure bookwork.

Question 5

[I proved this in a different way to how Dr. Zsak proved in the revision class]

Consider:

$$p(x) = \limsup_{n\to\infty} \frac{x_1 + \ldots + x_n}{n}$$

First note that:

$$\frac{x_1 + \ldots + x_n}{n} \leq \|x\|_\infty$$

for each n, so we have $p(x) \leq \|x\|$. Then, note that:

$$\sup_{k\geq n} \frac{(x_1+y_1) + \ldots + (x_n+y_n)}{n} \leq \sup_{k\geq n} \frac{x_1 + \ldots + x_n}{n} + \sup_{k\geq n} \frac{y_1 + \ldots + y_n}{n}$$

Taking $n \to \infty$ we get $p(x+y) \leq p(x) + p(y)$. Clearly we have positive homogeneity. Extending the zero functional we obtain L such that $Lx \leq p(x) \leq \|x\|$. Swapping x for $-x$ we have $|Lx| \leq \|x\|$. Hence $L \in \ell_\infty^*$. We prove each property.

(b): Let x be a convergent sequence. Then we have:

$$\begin{aligned} Lx &\leq \limsup_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right) \\ &\leq \limsup_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \sup_{j\geq k} x_j \right) \\ &= \lim_{n\to\infty} \sup_{j\geq n} x_j \\ &= \lim_{n\to\infty} x_n \end{aligned}$$

Similarly:

$$\begin{aligned} Lx &= -L(-x) \\ &\geq -\limsup_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} (-x_k) \right) \\ &= \liminf_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right) \\ &\geq \liminf_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \inf_{j\geq k} x_j \right) \\ &= \lim_{n\to\infty} \inf_{j\geq n} x_j \\ &= \lim_{n\to\infty} x_n \end{aligned}$$

So we have $Lx = \lim_{n\to\infty} x_n$.

(a): We already have $\|L\| \leq 1$, and since $L(1,1,\ldots,1) = \lim_{n\to\infty} 1 = 1$ while $\|(1,1,\ldots,1)\|_\infty = 1$, so $\|L\| = 1$.

(c): Let S be the shift operator on ℓ_1. We have:

$$L(x - Sx) \leq \limsup_{n\to\infty} \frac{(x_1 - x_2) + (x_2 - x_3) + \ldots + (x_{n-1} - x_n)}{n} = \limsup_{n\to\infty} \frac{x_1 - x_n}{n} = 0$$

since x is bounded. Similarly:

$$\begin{aligned}
L(x - Sx) &= -L(Sx - x) \\
&\geq -\limsup_{n\to\infty} \frac{(x_2 - x_1) + (x_3 - x_2) + \ldots + (x_n - x_{n-1})}{n} \\
&= \liminf_{n\to\infty} \frac{x_1 - x_n}{n} \\
&= 0
\end{aligned}$$

So we have $Lx = L(Sx)$, proving (a).