Question 1

Pure bookwork.

Question 2

[This solution is based off notes taken during Dr. Zsak's example class]

(iii) Since Y has finite codimension in X^* so there exists $\{e_1, \ldots, e_n\}$ such that $X^* = Y + \operatorname{span}\{e_1, \ldots, e_n\}$ where $Y \cap \operatorname{span}\{e_1, \ldots, e_n\} = 0$. Define:

$$\phi_j(f_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and $\phi_j(y) = 0$ for $y \in Y$. Then we have $\ker \phi_j = Y + \operatorname{span}\{e_i : i \neq j\}$. So $Y = \bigcap_{j=1}^n \ker \phi_j$.

Suppose that $x \in F \cap X$. If $x \in F$ then $Y \subseteq \ker \hat{x}$. But $\hat{x}$ is w^*–continuous, we have that $\ker \hat{x}$ is w^*–closed. But Y is w^*–dense, which implies that $\ker \hat{x} = X^*$. So $x = 0$.

We now want to show $d = d(S_X, F) > 0$. Let $\psi \in S_F$, (which is therefore not in X) then $d(\psi, X) > 0$ since X is closed in X^{**}. The map $\psi \mapsto d(\psi, x)$ defined on S_F is a continuous map on a compact space, so attains an infimum, so there exists $\delta > 0$ such that $d(\psi, X) \geq \delta$ for all $\psi \in S_F$.

Now take $\psi \in F$ general and $x \in S_X$. Then:

$$\|\psi - x\| = \|\psi\| \left\| \frac{\psi}{\|\psi\|} - \frac{x}{\|\psi\|} \right\| \geq \delta \|\psi\|$$

Hence if $\|\psi\| \geq 1/2$ we have $\|\psi - x\| \geq \delta/2$. For $\|\psi\| \leq 1/2$ we have $\|\psi - x\| \geq 1/2$. So we have $d \geq \max\{1/2, \delta\} > 0$.

Let $x \in S_X$ and let $E = \operatorname{span}(F \cup \{\hat{x}\})$. Define the map $\psi = 0$ on F and $\psi(\hat{x}) = 1$, then extend linearly. We have $d(x, F) \geq d$, so we must have $\|\psi\| \leq 1/d$. Extend using Hahn–Banach to get $\psi \in X^{***}$ vanishing on F. Then $\|d\psi\| \leq 1$, so there exists $f \in X^*$ with $\|f\| < 1 + \varepsilon$ such that $d\psi = \hat{f}$ on E. Then $\psi = \widehat{f/d}$ with $\|f/d\| < 1/d + \varepsilon/d$. Then we have:

$$\frac{\hat{f}}{\|\hat{f}\|}(x) = \frac{1}{\|\hat{f}\|} \geq \left(\frac{1}{d} + \frac{\varepsilon}{d}\right)^{-1} = d\,(1 + \varepsilon)^{-1}$$

Note that $\hat{f}$ restricts to 0 on F, so $f \in \bigcap_{i=1}^n \ker \phi_k = Y$. Hence we obtain <u>for $x \in S_X$</u> and each $\varepsilon > 0$:

$$\sup\{f(x) : f \in Y,\ \|f\| \leq 1\} \geq d(1 + \varepsilon)^{-1}$$

Hence:

$$\sup\{f(x) : f \in Y,\ \|f\| \leq 1\} \geq d$$

So for general non–zero $x \in X$ we have by scaling:

$$\sup\{f(x) : f \in Y,\ \|f\| \leq 1\} \geq d\,\|x\|$$

The demand is then immediate.

(iv): Let $f \in X^* \setminus \overline{Z}^{w^*}$. Then there exists a w^*–continuous map $\phi = \hat{x} \in X$ such that ϕ restricts to 0 on $\overline{Z}^{w^*}$ and is non–zero at f. Then we would have:

$$0 < c \|f\| \leq \sup \{g(x) : g \in S_Z\} = 0$$

since $\hat{x}$ vanishes on $\overline{Z}^{w^*}$ hence S_Z.

An example for the last part is $X = \ell_1$, $Z = c_0 \subseteq \ell_\infty$. We have for each $f \in Z$ and $x \in \ell_1$:

$$f(x) = \sum_{n=0}^{\infty} f_n x_n$$

for some $\langle f_n \rangle_{n \in \mathbf{N}} \in \ell_\infty$. Notice that we can pick $f_n = \operatorname{sgn}(x_n)$ (clearly defining a bounded sequence) to get $f(x_n) = \|x\|_1$. Hence we get the 1–norming property.

Question 3

[This solution is based off notes taken during Dr. Zsak's example class]

Bookwork up until last bit.

Since F is finite dimensional, S_F is norm compact. So there exists a cover by δ–balls with centres $\{f_1, \ldots, f_k\}$. Fix $\mu \in B_{C(K)^*}$. Let:

$$U = \left\{ \nu \in B_{C(K)^*} : |(\nu - \mu)(f_j)| < \delta \text{ for each } 1 \leq j \leq k \right\}$$

Since $B_{C(K)^*} = \overline{\operatorname{conv}\{\alpha\delta_k : k \in [0,1], |\alpha| = 1\}}^{w^*}$, U intersects $\operatorname{conv}\{\alpha\delta_k : k \in [0,1], |\alpha| = 1\}$. So there $n \in \mathbf{N}$, $|\alpha_j| = 1$, $w_j \in [0,1]$ and $s_j \geq 0$ $(1 \leq j \leq n)$ with $\sum_{j=1}^n s_j = 1$. Then we have:

$$\left| \int_0^1 f_i d\mu - \sum_{i=1}^n t_i f_i(w_i) \right| < \delta$$

where $t_i = s_i \alpha_i$ so that $|t_i| = s_i$. We now have for $f \in S_F$:

$$\begin{aligned} \left| \int_0^1 f d\mu - \sum_{i=1}^n t_i f(w_i) \right| &= \left| \int_0^1 (f - f_i) d\mu - \sum_{i=1}^n t_i (f - f_i)(w_i) \right| + \left| \int_0^1 f_i d\mu - \sum_{i=1}^n t_i f_i(w_i) \right| \\ &= \delta + \sum_{i=1}^n |t_i| + \delta \\ &= 3\delta \end{aligned}$$

Since this holds for all $f \in X^*$, we have:

$$\left| \int_0^1 f d\mu - \sum_{i=1}^n t_i f(w_i) \right| \leq 3\delta \|f\|$$

Taking $\delta = \varepsilon/3$ we are done.

Question 4

Bookwork up to invariant subspaces. Let $B(X)$ be the Banach algebra of bounded operators on X. Let A be the maximal commutative subalgebra of $B(X)$ that contains T. Then $\sigma_A(T) = \sigma(T)$. Since $\sigma(T)$ is disconnected, there exists disjoint closed sets C_1, C_2 such that $\sigma(T) = C_1 \cup C_2$. Since C_1, C_2 are disjoint and closed, we have $\varepsilon = d(C_1, C_2) > 0$. So let U and V be open neighborhoods of C_1 and C_2 that are still disjoint, say $U = C_1 + B_{\varepsilon/2}(0)$ and $V = C_2 + B_{\varepsilon/2}(0)$.

Then $U \cup V$ certainly contains $\sigma(T)$ and has precisely two connected components, U and V. Define $f : U \cup V$ by $f(z) = 1$ on U and $f(z) = 0$ on V. This is holomorphic since it is holomorphic on both connected components. Notice that $f^2 = 1$. Then consider $\Theta_T(f) = P$. This is a projection since $P^2 = \Theta_T(f^2) = \Theta_T(f) = P$. $P \notin \{0, I\}$ since $\sigma_A(\Theta_T(f)) = f(\sigma(T)) = \{0, 1\}$ by the HFC. Since A is a commutative algebra, we have $PT = TP$. Notice now that $Y = \ker(I - P) = \mathrm{im}(P)$ is a non–trivial closed subspace of X, and for each $y \in Y$ we have $Ty = TPy = PTy \in Y$. So Y is a non–trivial invariant subspace for T.

Question 5

Pure bookwork.